

A necessary and sufficient condition for a graph G , which satisfies the equality $\mu_{21}(G) = |V(G)|$

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Abstract

A necessary and sufficient condition is found for a graph G , which satisfies the equality $\mu_{21}(G) = |V(G)|$.

1 Introduction

We consider undirected, simple, finite and connected graphs, which contains at least one edge. The terms and concepts which are not defined can be found in [1]. The sets of vertices and edges of a graph G are denoted, respectively, by $V(G)$ and $E(G)$. The degree of a vertex $x \in V(G)$ is denoted by $d_G(x)$. Let $\Delta(G)$ be the maximum degree of a vertex of G . A function $\varphi : E(G) \rightarrow \{1, \dots, t\}$ is called a proper edge t -coloring of a graph G , if adjacent edges are colored differently and each of t colors is used. The least value of t , for which there exists a proper edge t -coloring of a graph G is denoted by $\chi'(G)$. For any graph G , and for any integer t , satisfying the inequality $\chi'(G) \leq t \leq |E(G)|$, we denote by $\alpha(G, t)$ the set of all proper edge t -colorings of G . Let:

$$\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{|E(G)|} \alpha(G, t).$$

An arbitrary nonempty finite subset of consecutive integers is called an interval. If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set of colors of edges incident

with x is denoted by $S_G(x, \varphi)$. If G is a graph, $\varphi \in \alpha(G)$, then $f_G(\varphi) \equiv |\{x \in V(G)/S_G(x, \varphi) \text{ is an interval}\}|$.

For a graph G and any integer t , we define [2]:

$$\mu_1(G, t) \equiv \min_{\varphi \in \alpha(G, t)} f_G(\varphi), \quad \mu_2(G, t) \equiv \max_{\varphi \in \alpha(G, t)} f_G(\varphi).$$

For any graph G , we set:

$$\begin{aligned} \mu_{11}(G) &\equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G, t), & \mu_{12}(G) &\equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G, t), \\ \mu_{21}(G) &\equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G, t), & \mu_{22}(G) &\equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G, t). \end{aligned}$$

Clearly, these parameters are correctly defined for an arbitrary graph. Some results on them were obtained in [2–15].

$\varphi \in \alpha(G)$ is called an interval edge coloring of a graph G if $f_G(\varphi) = |V(G)|$. The set of all graphs, for which there exists an interval edge coloring, is denoted by \mathfrak{N} . For any graph $G \in \mathfrak{N}$, we denote by $w(G)$ and $W(G)$, respectively, the least and the greatest value of t , for which G has an interval edge t -coloring.

For arbitrary integers n and i , satisfying the inequalities $n \geq 3$, $2 \leq i \leq n-1$, and for any sequence $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$ of nonnegative integers, we define the sets $V[i, A_{n-2}]$ and $E[i, A_{n-2}]$ as follows:

$$\begin{aligned} V[i, A_{n-2}] &\equiv \begin{cases} \{y_{i,1}, \dots, y_{i,a_{i-1}}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0, \end{cases} \\ E[i, A_{n-2}] &\equiv \begin{cases} \{(x_i, y_{i,j}) / 1 \leq j \leq a_{i-1}\}, & \text{if } a_{i-1} > 0 \\ \emptyset, & \text{if } a_{i-1} = 0. \end{cases} \end{aligned}$$

For any integer $n \geq 3$, and for any sequence $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$ of nonnegative integers, we define a graph $T[A_{n-2}]$ as follows:

$$\begin{aligned} V(T[A_{n-2}]) &\equiv \{x_1, \dots, x_n\} \cup \left(\bigcup_{i=2}^{n-1} V[i, A_{n-2}] \right), \\ E(T[A_{n-2}]) &\equiv \{(x_i, x_{i+1}) / 1 \leq i \leq n-1\} \cup \left(\bigcup_{i=2}^{n-1} E[i, A_{n-2}] \right). \end{aligned}$$

A graph G is called a galaxy, if either $G \cong K_2$, or there exist an integer $n \geq 3$ and a sequence $A_{n-2} \equiv (a_1, a_2, \dots, a_{n-2})$ of nonnegative integers, for which $G \cong T[A_{n-2}]$.

In this paper a necessary and sufficient condition is found for the equality $\mu_{21}(G) = |V(G)|$.

2 Preliminary Notes

Proposition 1. *For an arbitrary galaxy G the following statements hold:*

1. $G \in \mathfrak{N}$,
2. $w(G) = \Delta(G)$,
3. $W(G) = |E(G)|$,
4. for an arbitrary integer t , satisfying the inequality $w(G) \leq t \leq W(G)$, there exists $\varphi_t \in \alpha(G, t)$ with $f_G(\varphi_t) = |V(G)|$.

Proof follows from the results of [16] (the translation is available on <http://arxiv.org/abs/1308.2541v1>).

Proposition 2. [17–19] For any graph G , $\varphi \in \alpha(G, |E(G)|)$ with $f_G(\varphi) = |V(G)|$ exists iff G is a galaxy.

3 Main Result

Proposition 3. For arbitrary galaxy G , the equality $\mu_{21}(G) = |V(G)|$ is true.

Proof. Let G be an arbitrary galaxy. From proposition 1 it follows that for $\forall t \in [\Delta(G), |E(G)|]$, there exists $\varphi_t \in \alpha(G, t)$ with $f_G(\varphi_t) = |V(G)|$. It, particularly, means, that for $\forall t \in [\Delta(G), |E(G)|]$ $\mu_2(G, t) = |V(G)|$. It, by definition, means that $\mu_{21}(G) = |V(G)|$.

The proposition is proved.

Theorem 1. For any graph G , the equality $\mu_{21}(G) = |V(G)|$ holds if and only if G is a galaxy.

Proof. Let G be a galaxy. From proposition 3 it follows that $\mu_{21}(G) = |V(G)|$.

Let G be a graph with $\mu_{21}(G) = |V(G)|$. Let us show that G is a galaxy. Assume the contrary: G is not a galaxy. From proposition 2 we obtain that for $\forall \varphi \in \alpha(G, |E(G)|)$ $f_G(\varphi) \leq |V(G)| - 1$. It means that $\mu_2(G, |E(G)|) \leq |V(G)| - 1$. Now, taking into account the definition of the parameter μ_{21} , we conclude that $\mu_{21}(G) \leq |V(G)| - 1$. Contradiction.

The theorem is proved.

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